Circularly symmetric Green tensors for the harmonic vector wave equation in spheroidal coordinate systems

1978 J. Phys. A: Math. Gen. 11 749

(http://iopscience.iop.org/0305-4470/11/4/016)

J. Phys. A: Math. Gen., Vol. 11, No. 4, 1978. Printed in Great Britain. © 1978

Circularly symmetric Green tensors for the harmonic vector wave equation in spheroidal coordinate systems

D J N Wall†

Department of Electrical Engineering, University of Canterbury, Christchurch, 1, New Zealand

Received 26 September 1977

Abstract. Green tensor expansions for the vector harmonic wave equation, in spheroidal coordinate systems, are derived. The tensor expansions are suitable for application to problems in which the fields are circularly (rotationally) symmetric. Two expansions of the Green tensor are given; one appropriate to infinite free space and the other appropriate to the exterior of a spheroid having a vector Dirichlet boundary condition prescribed on its surface.

1. Introduction

Eigenfunction expansions of the free space Green function for the scalar Helmholtz equation have been found in many coordinate systems. Morse and Feshbach (1953, § 7.2) describe a method whereby the appropriate expansion can be found in any coordinate system for which the scalar Helmholtz equation is separable.

The Green function for the harmonic vector wave equation is commonly known in the engineering literature as the dyadic Green function. It is a tensor of rank two and we will refer to it as the Green tensor. The Green tensor is somewhat more elusive than its scalar counterpart, as a glance through the recent literature confirms (cf Howard 1974, Rahmat-Samii 1975, Tai and Rozenfeld 1976). Tai (1971) has provided a collection of Green tensor expansions for the harmonic vector wave equation in a number of coordinate systems. It should be noted that, as has recently been pointed out by Tai (1973), these expansions are only suitable when the field and source points do not coincide.

Recently while extending the generalised null field approach from scalar (Bates and Wall 1977) to vector diffraction problems (Wall 1976) the need arose for a free space Green tensor expansion in the spheroidal coordinate system. To our knowledge this tensor has not appeared in the literature, probably because the solution of the classic problem of radiation from a spheroid with circularly symmetric excitation does not need the deployment of a technique as powerful as the Green tensor technique.

In this paper we give a derivation of the free space Green tensor expansion and the Green tensor expansion satisfying a vector Dirichlet boundary condition, both in terms of the spheroidal coordinates and for circularly symmetric fields.

† Present address: Department of Mathematics, The University, Dundee, DD1 4HN, UK.

2. The vector eigenfunctions

To find a suitable eigenfunction expansion for the Green tensor it is first necessary to consider vector eigenfunctions, in the spheroidal coordinate system, of the harmonic vector wave equation

$$\nabla\times\nabla\times\boldsymbol{F}-k^2\boldsymbol{F}=0, \tag{1}$$

where the parameter k denotes the wavenumber. The partial differential equation (1) defines a vector field $\boldsymbol{F}$ existing in an infinite space Y in which a prolate spheroidal coordinate system (ξ, η, ϕ), at an origin O and with semi-focal distance d, has been defined. In all subsequent analyses only the prolate spheroidal coordinate system is considered although it is shown in § 3.3 how all the results derived here can be applied to the oblate spheroidal coordinate system by a simple transformation. The prolate spheroidal coordinates are related to a Cartesian coordinate system (x, y, z), also located at O, by the transformation

$$x=d[(1-\eta^2)(\xi^2-1)]^{1/2}\cos\phi, \qquad y=d[(1-\eta^2)(\xi^2-1)]^{1/2}\sin\phi, \qquad z=d\eta\xi, \tag{2}$$

with

$$-1\leqslant\eta\leqslant1, \qquad 1\leqslant\xi<\infty, \qquad 0\leqslant\phi\leqslant2\pi. \tag{3}$$

In the prolate spheroidal system the surface $\xi=\text{constant}>1$ is an elongated ellipsoid of revolution with major axis of length $2d\xi$, minor axis of length $2d(\xi^2-1)^{1/2}$ and eccentricity $1/\xi$.

All fields are to be taken as complex functions of space varying in time with angular frequency ω but with the time factor $\exp(\mathrm{i}\omega t)$ suppressed. Unit vectors in the various coordinate directions are denoted by the appropriate vector symbol with a circumflex accent and tensors will be signified by a vector symbol with an overbar.

The interest in this paper is in circularly symmetric fields; it is therefore convenient to assume that the field will be independent of the azimuthal angular variable ϕ. It seems that it is impossible to construct analytically a Green tensor for more general fields in the spheroidal coordinates because of the restricted separability of the vector Helmholtz equation (Spencer 1951, Spencer and Wells 1951).

A vector field $\boldsymbol{F}$ may always be uniquely separated into a longitudinal (irrotational) part $\boldsymbol{F}_\mathrm{l}$, and a transverse (solenoidal) part $\boldsymbol{F}_\mathrm{t}$ by the Helmholtz theorem (Morse and Feshbach 1953, § 1.5).

The Green tensor can also be decomposed into longitudinal and transverse parts. As the longitudinal part of the tensor is not of interest in the majority of applications, and its easiest method of derivation differs from that of the transverse part, it is only considered in appendix 1. Therefore only the transverse part of the vector eigenfunctions need be considered. All fields appearing subsequently in this paper, unless stated otherwise, should be considered as tranverse. The subscript t is therefore dropped from all symbols representing transverse vector fields for notational convenience.

The tranverse component of a vector field may always be derived from a pair of scalar fields. As is customary, we will denote the independent transverse vectors derived from these scalar fields by the symbols $\boldsymbol{M}$ and $\boldsymbol{N}$.

Choosing the vector eigenfunction $\boldsymbol{M}$ to be

$$\boldsymbol{M}=\hat{\boldsymbol{\phi}}\psi, \tag{4}$$

where ψ is a scalar function of the variables ξ and η, it may be shown that $\boldsymbol{M}$ is a solution of (1) provided that ψ is a solution of the equation

$$\left((\xi^2-1)\frac{\partial^2}{\partial\xi^2}+(1-\eta^2)\frac{\partial^2}{\partial\eta^2}+d^2k^2(\xi^2-\eta^2)\right)[d(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}\psi]=0. \tag{5}$$

The other independent tranverse vector eigenfunction is defined by

$$\boldsymbol{N}=\frac{1}{k}\nabla\times\boldsymbol{M}. \tag{6}$$

It may be easily verified that $\boldsymbol{N}$ given by (6) satisfies the vector differential equation (1), provided ψ is a solution of (5), and also satisfies the cyclic relation

$$\nabla\times\boldsymbol{N}=k\boldsymbol{M}. \tag{7}$$

$\boldsymbol{M}$ and $\boldsymbol{N}$ as defined in (4) and (6) are readily seen to be transverse. The particular solutions of (5) which are finite, continuous, and single-valued throughout Y form a discrete set. We shall denote any one of these solutions by ψ_n. Associated with each eigenfunction ψ_n are the vector solutions $\boldsymbol{M}_n$ and $\boldsymbol{N}_n$. The equation (5) is a particular form of the prolate spheroidal wave equation, and therefore the ψ_n can be written in terms of the prolate spheroidal eigenfunctions as

$$\psi_n=R^{(p)}_{1,n}(kd,\xi)S_{1,n}(kd,\eta), \qquad n=1,2,3\ldots. \tag{8}$$

In (8), $S_{1,n}(\,.\,)$ denotes the prolate spheroidal angle function of azimuthal index 1 and of order n. $R^{(p)}_{1,n}(\,.\,)$ denotes the prolate spheroidal radial function of the pth kind with azimuthal index 1 and of order n. The superscript p which denotes the kind of radial function takes on the values 1, 2, 3 and 4. These numbers are used to denote radial functions which correspond to eigenfunctions that represent respectively, standing waves which are regular at the origin, standing waves which are singular at the origin, and travelling waves which are incoming at infinity and travelling waves which are outgoing at infinity. Flammer (1957) and Meixner and Schäfke (1954) give detailed discussions on the properties of the spheroidal eigenfunctions. Use of (8) and (4) in (6) gives the full expression for the vector eigenfunction $\boldsymbol{N}_n$ as

$$\boldsymbol{N}^{(p)}(\xi,\eta;k)=(\xi^2-\eta^2)^{-1/2}\Big(S_{1,n}(kd,\eta)\frac{\mathrm{d}}{\mathrm{d}\xi}[(\xi^2-1)^{1/2}R^{(p)}_{1,n}(kd,\xi)]\hat{\boldsymbol{\eta}}$$
$$-R^{(p)}_{1,n}(kd,\xi)\frac{\mathrm{d}}{\mathrm{d}\eta}[(1-\eta^2)^{1/2}S_{1,n}(kd,\eta)]\hat{\boldsymbol{\xi}}\Big)(kd)^{-1}. \tag{9}$$

The superscript p is attached to the eigenfunctions $\boldsymbol{M}$ and $\boldsymbol{N}$ to denote the kind of radial function employed.

The vector eigenfunctions satisfy orthogonality relations which are important to our later development. These relations are considered in appendix 2 and are

$$\begin{aligned}
&\int_{\mathrm{Y}}\boldsymbol{M}^{(1)}_n(\xi,\eta;k)\,.\,\boldsymbol{N}^{(1)}_m(\xi,\eta;\kappa)\,\mathrm{d}v=0,\\
&\int_{\mathrm{Y}}\boldsymbol{N}^{(1)}_n(\xi,\eta;k)\,.\,\boldsymbol{N}^{(1)}_m(\xi,\eta;\kappa)\,\mathrm{d}v=\Lambda_{nn}\delta_{mn},\\
&\int_{\mathrm{Y}}\boldsymbol{M}^{(1)}_n(\xi,\eta;k)\,.\,\boldsymbol{M}^{(1)}_m(\xi,\eta;\kappa)\,\mathrm{d}v=\Lambda_{nn}\delta_{mn},
\end{aligned} \tag{10}$$

where δ_{mn} is the Kronecker delta.

3. Eigenfunction expansions of the transverse Green tensor

As explained in the previous section we are concerned only with the transverse part of the Green tensor here; the longitudinal part is considered separately in appendix 1.

The circularly symmetric Green tensor satisfies an inhomogeneous partial differential equation of the form

$$\nabla\times\nabla\times\bar{\boldsymbol{G}}-k^2\bar{\boldsymbol{G}}=\bar{\boldsymbol{D}}(\xi,\eta;\xi',\eta'), \tag{11}$$

where, as explained previously, we have omitted the subscript denoting that all the tensors in (11) are transverse. In (11) $\bar{\boldsymbol{D}}(\,.\,)$ is the tensor ring function. It is independent of ϕ and can be defined as a tensor which, when operating on any circularly symmetric vector field, say $\boldsymbol{F}(\xi',\eta')$, yields (on integrating over Υ in the primed coordinates) just the transverse part of $\boldsymbol{F}(\xi,\eta)$ (cf Morse and Feshbach 1953, § 13.1).

The completeness of the vector eigenfunctions discussed in § 2 in the space of piecewise continuous, circularly symmetric, transverse vector fields ensures that $\bar{\boldsymbol{D}}(\,.\,)$ can be written as

$$\bar{\boldsymbol{D}}(\xi,\eta;\xi',\eta')=\int_0^\infty\sum_{n=1}^\infty(\boldsymbol{M}_n^{(1)}(\xi,\eta;\kappa)\boldsymbol{A}_n(\xi',\eta';\kappa)+\boldsymbol{N}_n^{(1)}(\xi,\eta;\kappa)\boldsymbol{B}_n(\xi',\eta';\kappa))\,\mathrm{d}\kappa, \tag{12}$$

where the initially unknown posterior functions $\boldsymbol{A}_n(\,.\,)$ and $\boldsymbol{B}_n(\,.\,)$ are to be determined. Use of the relations (10) readily shows that

$$\begin{aligned}\boldsymbol{A}_n(\xi',\eta';\kappa)&=(\kappa/\pi)^2\boldsymbol{M}_n^{(1)}(\xi',\eta';\kappa)/I_{1,n},\\ \boldsymbol{B}_n(\xi',\eta';\kappa)&=(\kappa/\pi)^2\boldsymbol{N}_n^{(1)}(\xi',\eta';\kappa)/I_{1,n},\end{aligned} \tag{13}$$

where the normalising coefficient $I_{1,n}$ is discussed in appendix 2.

3.1. *Free space Green tensor*

Following the Ohm–Rayleigh procedure (Tai 1971) the free space Green tensor, which we now denote by $\bar{\boldsymbol{G}}_0$, is assumed to be of the form

$$\begin{aligned}\bar{\boldsymbol{G}}_0=\int_0^\infty\left(\frac{\kappa}{\pi}\right)^2\sum_{n=1}^\infty\frac{1}{I_{1,n}}&(\alpha_n\boldsymbol{M}_n^{(1)}(\xi,\eta;\kappa)\boldsymbol{M}^{(1)}(\xi',\eta';\kappa)\\ &+\beta_n\boldsymbol{N}_n^{(1)}(\xi,\eta;\kappa)\boldsymbol{N}_n^{(1)}(\xi',\eta';\kappa))\,\mathrm{d}\kappa.\end{aligned} \tag{14}$$

By substitution of (14) into (11) and use of (12) and (13) the unknown functions α_n and β_n can be determined as

$$\alpha_n=\beta_n=1/(\kappa^2-k^2). \tag{15}$$

The dependence on $R_{1,n}^{(1)}(\kappa d,\xi)R_{1,n}^{(1)}(\kappa d,\xi')$ of a tensor such as $\boldsymbol{M}_n^{(1)}(\xi,\eta;\kappa)\boldsymbol{M}_n^{(1)}(\xi',\eta';\kappa)$ can be written in the operational form

$$\boldsymbol{M}_n^{(1)}(\xi,\eta;\kappa)\boldsymbol{M}_n^{(1)}(\xi',\eta';\kappa)=\bar{\boldsymbol{T}}_n(R_{1,n}^{(1)}(\kappa d,\xi)R_{1,n}^{(1)}(\kappa d,\xi')), \tag{16}$$

where $\bar{\boldsymbol{T}}_n$ is some linear tensor operator.

An operational form of the integral identity derived in appendix 3, with $g(\kappa)=\kappa^2$ can then be written as

$$\int_0^{\infty} \frac{\kappa^2}{\kappa^2-k^2} \bar{T}_n (R_{1,n}^{(1)}(\kappa d, \xi) R_{1,n}^{(1)}(\kappa d, \xi')) \, d\kappa$$

$$= \begin{cases} -\dfrac{i\pi k}{2} M_n^{(4)}(\xi, \eta; k) M_n^{(1)}(\xi', \eta'; k), & \xi > \xi' \\ -\dfrac{i\pi k}{2} M_n^{(1)}(\xi, \eta; k) M_n^{(4)}(\xi', \eta'; k), & \xi' > \xi. \end{cases} \tag{17}$$

By repeating the same technique an operational integral relationship involving the $N_n(\,.\,)$ functions can be obtained. Equation (14), with α_n and β_n given by (15), can be simplified by use of the operational integral identity (17) and the corresponding equation involving the N_n eigenfunctions, to perform the κ integration. The expansion for the circularly symmetric free space Green tensor can then be written as

$$\bar{G}_0 = -\frac{ik}{2\pi} \sum_{n=1}^{\infty} \frac{1}{I_{1,n}} (M_n^{(1)}(\xi, \eta; k) M_n^{(4)}(\xi', \eta'; k)$$
$$+ N_n^{(1)}(\xi, \eta; k) N_n^{(4)}(\xi', \eta'; k)), \qquad \xi' > \xi. \tag{18}$$

The superscripts (1) and (4) in (18) are interchanged when $\xi > \xi'$.

3.2. *Green tensor satisfying the vector Dirichlet boundary condition*

We derive here the Green tensor, denoted by $\bar{G}_1$, which satisfies the vector Dirichlet boundary condition

$$\hat{n} \times \bar{G}_1(\xi, \eta; \xi', \eta') = 0, \qquad (\xi, \eta) \in S, \tag{19}$$

on the surface S of a spheroid of eccentricity $1/\xi_0$. In (19) $\hat{n}$ denotes the unit outward normal to S. In the preceding subsection a Green tensor possessing the correct singularity for a circularly symmetric ring source was given, but it will not satisfy (19). In order to comply with this boundary condition a suitable solution of the homogeneous form of (11), say $\bar{G}_{1s}$, must be added to $\bar{G}_0$ to obtain

$$\bar{G}_1 = \bar{G}_0 + \bar{G}_{1s}. \tag{20}$$

In view of the composition of $\bar{G}_0$ the tensor representing the scattered part must have the form

$$\bar{G}_{1s} = -\frac{ik}{2\pi} \sum_{n=1}^{\infty} \frac{1}{I_{1,n}} (c_n M_n^{(4)}(\xi, \eta; k) M_n^{(4)}(\xi', \eta'; k) + d_n N_n^{(4)}(\xi, \eta; k) N_n^{(4)}(\xi', \eta'; k)). \tag{21}$$

By making use of (18) through (21) and of the orthogonality of the vector eigenfunctions, and by noting $\hat{n} = \hat{\xi}$, we determine the unknown coefficients c_n and d_n in (21) to be

$$c_n = -\frac{R_{1,n}^{(1)}(kd, \xi_0)}{R_{1,n}^{(4)}(kd, \xi_0)}, \tag{22}$$

and

$$d_n = -\left(\frac{\mathrm{d}[(\xi^2-1)^{1/2}R^{(1)}_{1,n}(kd,\xi)]/\mathrm{d}\xi}{\mathrm{d}[(\xi^2-1)^{1/2}R^{(4)}_{1,n}(kd,\xi)]/\mathrm{d}\xi}\right)_{\xi=\xi_0}. \tag{23}$$

3.3. Expansion in the oblate spheroidal coordinate system

The analysis presented in this paper has been in terms of the prolate spheroidal coordinate system. The formulae derived are, however, suitable for the oblate system provided that the following simple transformations are made.

Terms involving $(\xi^2-\eta^2)$ and (ξ^2-1) become respectively $(\xi^2+\eta^2)$ and (ξ^2+1). The arguments of the spheroidal functions $S_{1,n}(.)$ and $R^{(p)}_{1,n}(.)$ become respectively $(-\mathrm{i}\kappa d, \eta)$ and $(-\mathrm{i}\kappa d, \mathrm{i}\xi)$.

4. Conclusions

In summary, we have derived two different Green tensors suitable for the spheroidal coordinates. That these Green tensors are useful in extending the generalised null field approach from scalar to vector diffraction problems we intend to show in later publications. The tensor $\bar{G}_1$ may be used to yield directly the solution to the classic problem of radiation from a spheroid with circularly symmetric excitation.

Acknowledgment

The support of a New Zealand University Grants Committee Postgraduate Scholarship is gratefully acknowledged.

Appendix 1. The longitudinal part of Green tensor

Howard (1974) has shown that the longitudinal part of the free space Green tensor satisfying the harmonic vector wave equation, for general fields, may be written in the following geometrically independent form

$$\bar{G}_l = -\frac{1}{4\pi k^2}\nabla\nabla' g, \tag{24}$$

where g is the scalar Green function for Laplace's equation, i.e.

$$g = 1/|\boldsymbol{x}-\boldsymbol{x}'|. \tag{25}$$

In (25) $\boldsymbol{x}$ and $\boldsymbol{x}'$ are the position vectors from O to the observation and source points respectively. The prime on the gradient operator in (24) indicates that it operates with respect to source coordinates.

For the special, circularly symmetric fields considered in this paper, $\bar{G}_l$ is again as in (24) but with the static Green function g replaced by g_0, where

$$g_0 = \frac{1}{2\pi}\int_0^{2\pi} g\,\mathrm{d}\phi. \tag{26}$$

Use of the static Green function expansion given by Morse and Feshbach (1953, § 10.3) enables us to write the longitudinal part of the circularly symmetric, free space Green tensor expansion in prolate spheroidal coordinates as

$$\bar{G}_l = -\frac{1}{4\pi k^2}\nabla\nabla'\Big(\sum_{n=0}^{\infty}(2n+1)P_n(\eta)P_n(\eta')P_n(\xi')Q_n(\xi)\Big), \qquad \xi>\xi' \quad (27)$$

where $P_n(\,.\,)$ and $Q_n(\,.\,)$ denote respectively the Legendre functions of the first and second kinds. The variables ξ and ξ' are interchanged in (27) when $\xi'>\xi$. The expansion analogous to (27) for the oblate spheroidal coordinate system is obtained by replacing ξ by $\mathrm{i}\xi$ and ξ' by $\mathrm{i}\xi'$ in the argument of the appropriate functions.

The longitudinal part of $\bar{G}_l$ may be readily found by similar procedures.

Appendix 2. Orthogonality of the vector eigenfunctions

From the definitions of the vector eigenfunctions (cf (4), (8) and (9)) it readily follows that

$$\int_Y \boldsymbol{M}_n^{(1)}(\xi,\eta;k)\,.\,\boldsymbol{N}_n^{(1)}(\xi,\eta;\kappa)\,\mathrm{d}v=0. \quad (28)$$

It is more involved to show the orthogonality of the $\boldsymbol{N}_n^{(1)}(\,.\,)$ eigenfunctions. However use of the definition (9), of the metric coefficients appropriate to the prolate spheroidal geometry (cf Flammer 1957, § 2.2), and of the defining equation (5), enable one to show after some manipulation that

$$\int_Y \boldsymbol{N}_n(\xi,\eta;k)\,.\,\boldsymbol{N}_m^{(1)}(\xi,\eta;\kappa)\,\mathrm{d}v=\int_Y \boldsymbol{M}_n^{(1)}(\xi,\eta;k)\,.\,\boldsymbol{M}_m^{(1)}(\xi,\eta;\kappa)\,\mathrm{d}v. \quad (29)$$

We define both integrals in (29) as equal to $\Lambda_{n,m}$. It is therefore sufficient to consider only the orthogonality of the $\boldsymbol{M}_n^{(1)}(\,.\,)$ eigenfunctions.

The spheroidal angle functions can be shown (Flammer 1957, § 3.1) to satisfy the orthogonality condition

$$\int_{-1}^{1} S_{1,n}(kd,\eta)S_{1,m}(kd,\eta)\,\mathrm{d}\eta=\delta_{nm}I_{1,n} \quad (30)$$

where $I_{1,n}$ is given by

$$I_{1,n}=I_{1,n}(kd)=\sum_{m=0}^{\infty}{}'\,(d_m^{1,n})^2\frac{2(m+2)}{(2m+3)m!}. \quad (31)$$

The coefficients $d_m^{1,n}$, which are functions of kd, can be determined via the differential equation (5) (Flammer 1957, chap. 3). The prime over the summation sign in (31) indicates that only even values of m are included if n is odd and only odd values of m are included if n is even.

In order to obtain the orthogonality properties of the $\boldsymbol{M}_n^{(1)}(\,.\,)$ eigenfunctions we consider a spherical polar coordinate system (r,θ,ϕ) superimposed on the spheroidal coordinate system at the origin O. The prolate spheroidal eigenfunctions can then be expressed in terms of the spherical eigenfunctions (see Flammer 1957, § 5.3) as

$$R^{(1)}_{1,n}(kd,\xi)S_{1,n}(kd,\eta)=-\sum_{m=0}^{\infty}{}' d^{1,n}_m(kd)j_n(kr)\frac{\partial P_{m+1}(\cos\theta)}{\partial\theta}, \qquad r>d, \tag{32}$$

where $j_n(\,.\,)$ denotes the spherical Bessel function of the first kind of order n.

Examination of the functional form of the $\boldsymbol{M}^{(1)}_n(\,.\,)$ eigenfunctions and use of (30) shows that $\Lambda_{n,m}=0$ unless $m=n$, so it will be sufficient to consider the form of $\Lambda_{n,n}$.

Use of the definition of the $\boldsymbol{M}^{(1)}_n(\,.\,)$ eigenfunction, equation (32) and the orthogonality of the Legendre functions (Morse and Feshbach 1953, chap. 10) enables Λ_{nn} to be reduced to

$$\Lambda_{n,n}=4\pi\sum_{m=0}^{\infty}{}'(d^{1,n}_m(kd))^2\frac{(m+2)!}{(2m+3)m!}\int_0^{\infty}j_{m+1}(\kappa r)j_{m+1}(kr)r^2\,\mathrm{d}r. \tag{33}$$

Hence the integral relationship

$$\frac{2}{\pi}\int_0^{\infty}j_n(\kappa x)j_n(\kappa x')x^2\,\mathrm{d}x=\delta(x-x')/x^2, \tag{34}$$

when combined with (31) allows (33) to be written as

$$\Lambda_{n,n}=\frac{\pi^2\delta(k-\kappa)}{k^2}I_{1,n}. \tag{35}$$

Appendix 3. An integral identity

The integral

$$I(\xi,\xi')=\int_0^{\infty}\frac{g(\kappa)}{\kappa^2-k^2}R^{(1)}_{m,n}(\kappa d,\xi)R^{(1)}_{m,n}(\kappa d,\xi')\,\mathrm{d}\kappa, \tag{36}$$

where $g(\kappa)$ is an even analytic function of κ, is evaluated here. Use of

$$R^{(1)}_{m,n}(\kappa d,\xi)=\tfrac{1}{2}(R^{(3)}_{m,n}(\kappa d,\xi)+R^{(4)}_{m,n}(\kappa d,\xi)), \tag{37}$$

in (36) allows its right-hand side to be written as the sum of two integrals. It is convenient to examine the integral involving $R^{(3)}_{m,n}(\,.\,)$ first; this integral is

$$I_1(\xi,\xi')=\frac{1}{2}\int_0^{\infty}\frac{g(\kappa)}{\kappa^2-k^2}R^{(1)}_{m,n}(\kappa d,\xi')R^{(3)}_{m,n}(\kappa d,\xi)\,\mathrm{d}\kappa, \tag{38}$$

where it is initially assumed that $\xi>\xi'$. With the change of variable $\kappa'=\mathrm{e}^{\mathrm{i}\pi}\kappa$ and taking note of the following (Meixner and Schäfke 1954, § 3.65):

$$\begin{aligned}
&R^{(p)}_{m,n}(\kappa d\,\mathrm{e}^{\mathrm{i}\pi},\xi)=R^{(p)}_{m,n}(\kappa d,\xi\,\mathrm{e}^{\mathrm{i}\pi}),\\
&R^{(3)}_{m,n}(\kappa d,\xi\,\mathrm{e}^{\mathrm{i}\pi})=\mathrm{e}^{-\mathrm{i}\pi}R^{(4)}_{m,n}(\kappa d,\xi),\\
&R^{(1)}_{m,n}(\kappa d,\xi\,\mathrm{e}^{\mathrm{i}\pi})=\mathrm{e}^{\mathrm{i}\pi}R^{(1)}_{m,n}(\kappa d,\xi),
\end{aligned} \tag{39}$$

(38) can be written as

$$I_1(\xi,\xi')=\frac{1}{2}\int_{-\infty}^{0}\frac{g(\kappa)}{\kappa^2-k^2}R^{(1)}_{m,n}(\kappa d,\xi')R^{(4)}_{m,n}(\kappa d,\xi)\,\mathrm{d}\kappa, \qquad \xi>\xi'. \tag{40}$$

Combining (40) with the second integral involving $R^{(4)}_{m,n}(\,.\,)$ obtained from (36) by use of (37) yields

$$I(\xi,\xi')=\frac{1}{2}\int_{-\infty}^{\infty}\frac{g(\kappa)}{\kappa^2-k^2}R^{(1)}_{m,n}(\kappa d,\xi')R^{(4)}_{m,n}(\kappa d,\xi)\,\mathrm{d}\xi,\qquad \xi>\xi'. \tag{41}$$

The integral in (41) can be evaluated by allowing κ to take on complex values and integrating along a contour in the κ-plane. The contour is along the real axis indented above the pole $\kappa=k$ and below the pole $\kappa=-k$ and closed by a large semi-circle in the lower half plane.

It is easy to show that the contribution from the large semi-circle vanishes in the limit as its radius becomes infinite when $\xi>\xi'$; the integral (41) is then equal to $2\pi i$ times its residue at the pole $\kappa=k$. When $\xi'>\xi$, $R^{(1)}_{m,n}(\kappa d,\xi')$ in (36) is replaced using (37) and a similar procedure to the above is followed to evaluate the resulting integral. Thus (36) becomes

$$I(\xi,\xi')=\begin{cases}-\dfrac{i\pi}{2k}g(k)R^{(4)}_{m,n}(kd,\xi)R^{(1)}_{m,n}(kd,\xi'), & \xi>\xi',\\[2ex] -\dfrac{i\pi}{2k}g(k)R^{(1)}_{m,n}(kd,\xi)R^{(4)}_{m,n}(kd,\xi'), & \xi'>\xi.\end{cases} \tag{42}$$

References

Bates R H T and Wall D J N 1977 *Phil. Trans. R. Soc.* A **287** 45
Flammer C 1957 *Spheroidal Wave Functions* (Stanford: Stanford University Press)
Howard A Q 1974 *Proc. IEEE* **62** 1704
Meixner J and Schäfke F W 1954 *Mathieusche Funktionen und Sphäroidfunctionen* (Berlin: Springer)
Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw-Hill)
Rahmat-Samii Y 1975 *IEEE Trans. Microwave Theory Tech.* **MTT-23** 762
Spencer D E 1951 *J. Appl. Phys.* **22** 386
Spencer D E and Wells C P 1951 *Commun. Pure Appl. Math.* **4** 95
Tai C T 1971 *Dyadic Green's Functions in Electromagnetic Theory* (Scranton, Pa: International Textbook)
—— 1973 *Proc. IEEE* **61** 480
Tai C T and Rozenfeld P 1976 *IEEE Trans. Microwave Theory Tech.* **MTT-24** 597
Wall D J N 1976 *PhD Thesis* University of Canterbury, Christchurch, New Zealand